

Weinberg's energy-momentum pseudotensor for Schwarzschild field

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Summary

Weinberg's energy-momentum pseudotensor is obtained for Schwarzschild metric in harmonic coordinates. On the horizon it possesses unintegrable singularities. For this reason the total energy of a collapsar can't be obtained by integrating energy density over the system's volume. The implication for gravity theories is noted. A thought on how to choose unique energy-momentum tensor is given.

1 Introduction

There is no energy-momentum tensor of gravitational field in general relativity. Instead there are infinitely many pseudotensors [1,2]. The concept of nonlocalizability of gravitational energy density was introduced to explain this unusual situation. Yet this concept seems unnecessary, if the gravity theory is build by field-theoretical means without requiring general covariance or if one assumes the existence of privileged frame. In general relativity the harmonic coordinates provide natural candidacy for privileged coordinates for an isolated system [3].

Weinberg's pseudotensor (the suffix pseudo is dropped in the following) is singled out by the fact that it is the source of gravitational field [4]. For this reason it is interesting to find it in harmonic coordinates, which goes over to Minkovskiiian ones far away from gravitating body. Due to unusual properties of space-time beyond horizon and interchanged role of time and radial coordinates there [5], we expect that it will be impossible to get the collapsar total energy by integrating energy density over system's volume. Calculations confirm this: the energy-momentum tensor have unintegrable singularities on the horizon.

2 Calculation of energy-momentum tensor

We use on the whole Weinberg notation, but denote harmonic coordinates x_i , $|\vec{x}| = r$ with small letters. Indices of $h_{\mu\nu}$, $R_{\mu\nu}^{(1)}$, $\frac{\partial}{\partial x^\mu}$ are raised and lowered with the help of η , indices of generally covariant tensors such as $R_{\mu\nu}$ are raised and lowered with the help of g . Latin indices run from 1 to 3.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \quad \phi = -\frac{GM}{r}, \quad (1)$$

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = \frac{1+\phi}{1-\phi}dt^2 - (1-\phi)^2 d\vec{x}^2 - \frac{1-\phi}{1+\phi}\phi^2 \frac{(\vec{x}d\vec{x})^2}{r^2}. \quad (2)$$

Now we reproduce eqs. (7.6.3)–(7.6.4) from [4]. The exact Einstein equations are written there in the form

$$R_{\mu\kappa}^{(1)} - \frac{1}{2}\eta_{\mu\kappa}R^{(1)} = -8\pi G[T_{\mu\kappa} + t_{\mu\kappa}], \quad (3)$$

where

$$t_{\mu\kappa} = \frac{1}{8\pi G}[R_{\mu\kappa} - \frac{1}{2}g_{\mu\kappa}R - R_{\mu\kappa}^{(1)} + \frac{1}{2}\eta_{\mu\kappa}R^{(1)}], \quad (4)$$

and $R_{\mu\kappa}^{(1)}$ is linear in h part of $R_{\mu\kappa}$:

$$R_{\mu\kappa}^{(1)} = \frac{1}{2}[h_{,\mu\kappa} - h^\lambda_{\mu,\lambda\kappa} - h^\lambda_{\kappa,\lambda\mu} + h_{\mu\kappa,\lambda}{}^\lambda], \quad h_{,\mu} = \frac{\partial}{\partial x^\mu}h. \quad (5)$$

Eq.(3) has the form of wave equation for spin-2 field. Its source is $T_{\mu\kappa} + t_{\mu\kappa}$. Hence, $t_{\mu\kappa}$ (i.e. energy-momentum tensor of gravitational field) is also a source of gravitational field. Eq. (3) is suggested by solution of Einstein equations by iteration. In linear approximation $h_{\mu\nu} = h^{(1)\mu\nu}$ is generated by material tensor. Inserting this solution of linearized equation in (4) and retaining quadratic in $h^{(1)}$ terms, we get $t_{\mu\kappa}^{(2)}$. [See eq.(7.6.14) in [4], in which it is not indicated explicitly that figuring there $h_{\mu\nu}$ are $h_{\mu\nu}^{(1)}$. The expression for $t_{\mu\nu}^{(2)}$, obtained from that eq., for Newtonian center is given in [6]. It coincides with eqs. (11), (13) below, which were found from exact expressions.] Further, $t_{\mu\nu}^{(2)}$ according to wave equation generates $h_{\mu\nu}^{(2)}$ and so on. The sum over all approximations gives $h_{\mu\kappa}$, which exactly satisfies Einstein equations.

Now we assume up to eq.(14) that the matter is absent in the considered region. Then the energy-momentum tensor has the form

$$t_{\mu\nu} = \frac{1}{8\pi G}[\frac{1}{2}\eta_{\mu\nu}R^{(1)} - R_{\mu\nu}^{(1)}], \quad R^{(1)} = R_\lambda^{(1)\lambda} = h_{,\lambda}{}^\lambda - h^{\mu\nu}{}_{,\mu\nu}, \quad h = h_\lambda{}^\lambda. \quad (6)$$

From (1-2) and (5-6) we find

$$\begin{aligned} h &= 2\phi^2 - 4\phi - 4 + \frac{2}{1-\phi} + \frac{2}{1+\phi}, \quad h_{,\lambda}{}^\lambda = \frac{4}{r^2}\phi^2[\frac{1}{(1-\phi)^3} + \frac{1}{(1+\phi)^3} + 1], \\ h_{ij} &= (1-\phi)^2\delta_{ij} + \frac{\phi^2 - \phi^3}{1+\phi}\frac{x_i x_j}{r^2} - \delta_{ij}, \\ h_{ij,kl} &= \frac{2x_i x_j x_k x_l}{r^6} \left(-12\phi^2 + 15\phi - 8 + \frac{3}{1+\phi} + \frac{3}{(1+\phi)^2} + \frac{2}{(1+\phi)^3} \right) + \\ &\quad \frac{2\delta_{ij}x_k x_l}{r^4}(4\phi^2 - 3\phi) + \frac{2\delta_{ij}\delta_{kl}}{r^2}(\phi - \phi^2) + \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{r^2} \left(-\phi^2 + 2\phi - 2 + \frac{2}{1+\phi} \right) \end{aligned} \quad (7)$$

$$\begin{aligned}
& + \frac{2}{r^4} [x_i x_j \delta_{kl} + x_k x_j \delta_{il} + x_i x_k \delta_{jl} + x_j x_l \delta_{ik} + x_i x_l \delta_{jk}] \times \\
& \times \left(2\phi^2 - 3\phi + 2 - \frac{1}{1+\phi} - \frac{1}{(1+\phi)^2} \right). \tag{8}
\end{aligned}$$

With the help of this expressions we obtain

$$\begin{aligned}
R_{kl}^{(1)} &= \frac{x_k x_l}{r^4} \left(2\phi^2 - 2 + \frac{1}{1+\phi} + \frac{1}{(1+\phi)^2} - \frac{1}{1-\phi} - \frac{1}{(1-\phi)^2} + \frac{2}{(1-\phi)^3} \right) + \\
& \quad \frac{\delta_{kl}}{r^2} \left(2 - \frac{3}{1+\phi} + \frac{1}{(1+\phi)^2} + \frac{1}{1-\phi} - \frac{1}{(1-\phi)^2} \right), \\
R^{(1)} &= \frac{1}{r^2} \left(2\phi^2 + 4 + \frac{4}{1-\phi} - \frac{8}{(1-\phi)^2} + \frac{4}{(1-\phi)^3} - \frac{8}{1+\phi} + \frac{4}{(1+\phi)^2} \right), \tag{9} \\
t_{kl} &= \frac{1}{8\pi G} \left[\frac{x_k x_l}{r^4} \left(-2\phi^2 + 2 - \frac{1}{1+\phi} - \frac{1}{(1+\phi)^2} + \frac{1}{1-\phi} + \frac{1}{(1-\phi)^2} - \frac{2}{(1-\phi)^3} \right) + \right. \\
& \quad \left. \frac{\delta_{kl}}{r^2} \left(\phi^2 - \frac{1}{1+\phi} + \frac{1}{(1+\phi)^2} + \frac{1}{1-\phi} - \frac{3}{(1-\phi)^2} + \frac{2}{(1-\phi)^3} \right) \right]. \tag{10}
\end{aligned}$$

For $\phi \ll 1$ we have

$$t_{ik}|_{\phi \ll 1} = \frac{\phi^2}{8\pi G} \left[\frac{7\delta_{ik}}{r^2} - \frac{14x_i x_k}{r^4} \right]. \tag{11}$$

Similarly, we find

$$t_{00} = \frac{1}{8\pi G r^2} \left(\frac{4}{1+\phi} - \frac{2}{(1+\phi)^2} - \phi^2 - 2 \right), \tag{12}$$

$$t_{00}|_{\phi \ll 1} = -\frac{3}{8\pi G} (\nabla\phi)^2 = -\frac{3GM^2}{8\pi r^4}. \tag{13}$$

Now we check that conservation laws $t^{\mu\nu}{}_{,\nu} = 0$ are fulfilled. As $t_{i0} = 0$, we need to verify that $t_{ni,n} = 0$. Straightforward calculation gives

$$\begin{aligned}
R_{ni,n}^{(1)} &= \frac{1}{2} R_{,i}^{(1)} = \\
& \frac{x_i}{r^4} \left[-4 - 4\phi^2 + \frac{4}{1+\phi} + \frac{4}{(1+\phi)^2} - \frac{4}{(1+\phi)^3} - \frac{2}{1-\phi} - \frac{2}{(1-\phi)^2} + \frac{10}{(1-\phi)^3} - \frac{6}{(1-\phi)^4} \right],
\end{aligned}$$

Q.E.D.

Introducing tensor

$$Q^{\rho\nu\lambda} = \frac{1}{2} [h^{,\nu} \eta^{\rho\lambda} - h^{,\rho} \eta^{\nu\lambda} - h^{\mu\nu}{}_{,\mu} \eta^{\rho\lambda} + h^{\mu\rho}{}_{,\mu} \eta^{\nu\lambda} + h^{\nu\lambda}{}_{,\rho} - h^{\rho\lambda}{}_{,\nu}], \tag{14}$$

with property $Q^{\rho\nu\lambda} = -Q^{\nu\rho\lambda}$, we have, see. Ch.7, §6 [4]:

$$Q^{\rho\nu\lambda}{}_{,\rho} = R^{(1)\nu\lambda} - \frac{1}{2} \eta^{\nu\lambda} R^{(1)} = -8\pi G \tau^{\nu\lambda}, \tag{15}$$

$$\tau^{\nu\lambda} = \eta^{\mu\nu}\eta^{\lambda\kappa}[T_{\mu\kappa} + t_{\mu\kappa}].$$

Due to this relation for smooth tensor $Q^{\rho\nu\lambda}$ the integral of total (gravitational and material) energy density over the volume of system may be written as an integral over remote surface (see [4])

$$P^0 = -\frac{1}{8\pi G} \int Q^{i00}{}_{,i} d^3x = -\frac{1}{8\pi G} \int Q^{i00} n_i r^2 d\Omega = M, \quad (16)$$

$$Q^{i00} = \frac{1}{2}(h_{jj,i} - h_{ij,j}) = \frac{x_i}{r^2} \left(2 - \phi^2 - \frac{2}{1+\phi} \right),$$

n_i are components of external normal to the surface. Yet in case of horizon we see from (10) and (12) that at $\phi = -1$ (i.e. at $r = GM$ in harmonic system) tensor $t_{\mu\kappa}$ has unintegrable singularity. This prevent us from using Gauss theorem (i.e. from going to the second equation in (16)) in the whole volume of system. But it is easy to find the energy outside a sphere of radius $r_1 = GM(1 + \delta)$:

$$\Delta P^0 = -\frac{r}{2G} \left(2 - \phi^2 - \frac{2}{1+\phi} \right) \Big|_{r_1}^{\infty} \quad (17)$$

For $0 < \delta \ll 1$ we get

$$\Delta P^0 = -M \left(\frac{1}{\delta} + \frac{1}{2} \right). \quad (19)$$

It is interesting to compare (19) with Dehnen result [7]. Using standard Schwarzschild coordinates ($r' = r + GM$) Dehnen find for his tensor

$$\Delta P^0 = -GM^2 \int_{r'_1}^{\infty} \frac{dr'}{r'^2 \left(1 - \frac{r_g}{r'} \right)^{\frac{3}{2}}} = M \left(1 - \frac{1}{\sqrt{1 - \frac{r_g}{r'_1}}} \right). \quad (20)$$

This expression has square root divergence for $r'_1 \rightarrow r_g$.

It is interesting that in Box 23.1 in [1] arguments are given in favor of localizability of gravitational energy density in case of spherical symmetry. According to this arguments the gravitational energy outside the matter ball is zero. In Newtonian limit this way of accountig for gravitational energy corresponds to accounting for gravitational attractions between different parts of matter. It is not clear how to reconcile this with (19) or (20) and with the concept that nonlinear corrections to gravitational field are generated by gravitational energy-momentum tensor. As to the nonuniqueness of energy-momentum tensors, one may note that if some tensor correctly describes the interaction with gravitons then it is natural to consider this tensor as the right one.

Although Schwarzschild singularity is considered fictitious (from the time of Lemaître, see box 31.1 in [1]), it is difficult to be reconciled with this. Physically the singularity manifest itself in impossibility for cosmonaut, crossing it, to return back, in unlimited growth of acceleration (in static frame) of freely falling

particle nearing horizon, in absence of static frame beyond horizon and in many other unusual things.

One can consider these singularities as a hint that in the regime of strong field the theory will be modified in the future and no drastic changes in space-time topology will be needed.

The author is grateful to V.I.Ritus for useful discussions and constructive remarks.

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